Solutions to H. W. #10

1. $\frac{du}{dt} = 4x^3(-\sin(t)) - 4y^3\cdot(2t\cos(t^2)) = -4\left(\cos^3(t)\sin(t) + \right.$

$\left. + 2t\sin^3(t^2)\cos(t^2)\right)$

2. $\frac{du}{dt} = e^{x+4y-2}\left(\frac{1}{t}\right) + 4e^{x+4y-2}\left(\frac{1}{t+1} - \frac{1}{t}\right) - e^{x+4y-2}\left(-\frac{1}{t^2}\right) =$

$= e^{x+4y-2}\left(\frac{1}{t} + \frac{4}{t+1} - \frac{4}{t} + \frac{1}{t^2}\right) = e^{\ln t + 4\ln(\frac{t+1}{t}) - \frac{1}{t}}\left(\frac{4}{t+1} - \frac{3}{t} + \frac{1}{t^2}\right)$

3. $\frac{du}{dt} = \frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}\frac{1}{2\sqrt{t}} + \frac{\frac{1}{x}}{1+(\frac{y}{x})^2}\cdot\frac{3}{2}\sqrt{t} =$

$= \frac{-y}{x^2+y^2}\frac{1}{2\sqrt{t}} + \frac{x}{x^2+y^2}\cdot\frac{3}{2}\sqrt{t} = \frac{-(\sqrt{t})^3}{t+t^3}\frac{1}{2\sqrt{t}} + \frac{\sqrt{t}}{t+t^3}\cdot\frac{3}{2}\sqrt{t} =$

$= \frac{1}{2}\left(\frac{-t}{t+t^3} + \frac{3t}{t+t^3}\right) = \frac{t}{t+t^3} = \frac{1}{1+t^2}$

4. $\frac{du}{dt} = \frac{\partial f}{\partial x}\cos t - \frac{\partial f}{\partial y}\sin t + \frac{\partial f}{\partial z}$

5. $\frac{du}{dt} = \frac{-2y}{(x-y)^2}f'(t) + \frac{2x}{(x-y)^2}g'(t) = \frac{-2g(t)}{(f(t)-g(t))^2}f'(t) + \frac{2f(t)}{(f(t)-g(t))^2}g'(t)$

6. $\frac{\partial u}{\partial t} = (2xy^3+1)\cdot 2t + (3x^2y^2-3)\cdot 1 \Big|_{x=t^2-s,\ y=t-s^2}.$

$\frac{\partial u}{\partial s} = (2xy^3+1)(-1) + (3x^2y^2-3)(-2s)\Big|_{x=t^2-s,\ y=t-s^2}.$

7. $\frac{\partial u}{\partial t} = 2+1-1 = 2 \qquad \frac{\partial u}{\partial s} = 3-1-1 = 1.$

8. $\frac{du}{dt} = \frac{(x^2+y^2)-(x+1)2x}{(x^2+y^2)^2} + \frac{-2(x+1)y}{(x^2+y^2)^2}(-2t)\Big|_{x=s^2+t,\ y=s-t^2}$

$\frac{du}{ds} = \frac{(x^2+y^2)-(x+1)2x}{(x^2+y^2)^2}\cdot 2s + \frac{-2(x+1)y}{(x^2+y^2)^2}\Big|_{x=s^2+t,\ y=s-t^2}$

9. $\frac{\partial u}{\partial t} = \frac{1}{x} + \frac{1}{y}(-3) + \frac{1}{z}\left(\frac{1}{(s-t)^2}\right) = \frac{1}{s+t} - \frac{3}{s-3t} + \frac{1}{(s-3t)^3}$

$\frac{\partial u}{\partial s} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\left(\frac{-1}{(s-t)^2}\right) = \frac{1}{s+t} + \frac{1}{s-3t} - \frac{1}{(s-t)^3}$

10. $\frac{\partial u}{\partial t} = \frac{\partial g}{\partial x}(-2t) + \frac{\partial g}{\partial y}(2s)$

$\frac{\partial u}{\partial s} = \frac{\partial g}{\partial x}(2s) + \frac{\partial g}{\partial y}(2t)$

11. $\frac{\partial u}{\partial t} = \frac{-s\,(2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t))}{(x^2(t)+y^2(t)+z^2(t))^2} =$

$= \frac{-2s\,(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))}{(x^2(t)+y^2(t)+z^2(t))^2}$

$\frac{\partial u}{\partial s} = \frac{1}{x^2(t)+y^2(t)+z^2(t)}$

12. $\frac{\partial u}{\partial t} = f(s+t, 2s+3t) + t\left(\frac{\partial f}{\partial x}(s+t, 2s+3t) + 3\frac{\partial f}{\partial y}(s+t, 2s+3t)\right) -$

$-s\left(3\frac{\partial f}{\partial x}(2s+3t, s+t) + \frac{\partial f}{\partial y}(2s+3t, s+t)\right)$

$\frac{\partial u}{\partial s} = t\left(\frac{\partial f}{\partial x}(s+t, 2s+3t) + 2\frac{\partial f}{\partial y}(s+t, 2s+3t)\right) - f(2s+3t, s+t) -$

$-s\left(2\frac{\partial f}{\partial x}(2s+3t, s+t) + \frac{\partial f}{\partial y}(2s+3t, s+t)\right).$

13. $\frac{\partial u}{\partial t} = e^t g(e^2+e^{-s}, \ln t, \ln(e^{t+s}+1)) + e^t\left(\frac{\partial g}{\partial x}\cdot 0 + \frac{\partial g}{\partial y}\cdot\frac{1}{t} +\right.$

$\left. + \frac{\partial g}{\partial z}\cdot\frac{e^{t+s}}{e^{t+s}+1}\right)$

$$\frac{\partial u}{\partial s} = e^{t}\left(\frac{\partial g}{\partial x}\cdot(-e^{-s})\right)$$

14, $f(3,2) = (-2,3)$.

$$Jg(-2,3) = \begin{pmatrix} 2xy^3 & 3x^2y^2 \\ 3 & -2y \end{pmatrix}\Bigg|_{(-2,3)} = \begin{pmatrix} -108 & 108 \\ 3 & -6 \end{pmatrix}$$

$$Jf(3,2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus $J(g\circ f)(3,2) = Jg(-2,3)\,Jf(3,2) = \begin{pmatrix} -108 & 108 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$

$= \begin{pmatrix} 108 & 108 \\ -6 & -3 \end{pmatrix}$ Hence $D(g\circ f)(3,2)(x,y) = (108x+108y, -6x-3y)$

15. $f(3,1,-1) = (-3, -\frac{1}{3})$.

$$Jg(-3,-\tfrac{1}{3}) = \begin{pmatrix} 2xy^3 & 3x^2y^2 \\ 3 & -2y \end{pmatrix}\Bigg|_{(-3,-\frac{1}{3})} = \begin{pmatrix} \frac{2}{9} & 3 \\ 3 & \frac{2}{3} \end{pmatrix}$$

$$Jf(3,1,-1) = \begin{pmatrix} z & 0 & x \\ \frac{-yz}{(xz)^2} & \frac{1}{xz} & \frac{-yx}{(xz)^2} \end{pmatrix}\Bigg|_{(3,1,-1)} = \begin{pmatrix} -1 & 0 & 3 \\ \frac{1}{9} & -\frac{1}{9} & \frac{-1}{3} \end{pmatrix}$$

$$J(g\circ f)(3,1,-1) = \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{-79}{27} & \frac{-2}{27} & \frac{79}{9} \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 3 & -9 & -9 \\ -79 & -2 & 237 \end{pmatrix}$$

Hence $D(g\circ f)(3,1,-1)(x,y,z) = \frac{1}{27}(3x-9y-9z, -79x-2y+2372)$

16. $f(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{6}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2})$

$$Jg(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, \tfrac{1}{2}) = \begin{pmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{pmatrix}\Bigg|_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2})} =$$

$$= \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

$$Jf(\tfrac{\pi}{4}, -\tfrac{\pi}{4}, \tfrac{\pi}{6}) = \begin{pmatrix} \cos x & 0 & 0 \\ 0 & -\sin y & 0 \\ \cos(x+y+z) & \cos(x+y+z) & \cos(x+y+z) \end{pmatrix}\Bigg|_{(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{6})} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Thus $J(g\circ f)(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{6}) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} =$

$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & 0 \end{pmatrix}$. Therefore $D(g\circ f)(\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{6})(x,y,z) =$

$= (y, \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}y + \frac{\sqrt{3}}{2}z, x)$

17. $f(0,1,0) = (1,0)$

$$Jg(1,0) = \begin{pmatrix} 3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}\Bigg|_{(1,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Jf(0,1,0) = \begin{pmatrix} 4 & 1 & 2x_3 \\ x_3 & 0 & x_1 \end{pmatrix}\Bigg|_{(0,1,0)} = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J(g \circ f)(0,1,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $D(g \circ f)(0,1,0)(x_1, x_2, x_3) = (12x_1 + 3x_2, 0)$

18. $$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial t} = \frac{-2x}{(x^2+y^2+z^2)^2}\cdot(6t-1) - \frac{2y}{(x^2+y^2+z^2)^2}\cdot 2t - \frac{2z}{(x^2+y^2+z^2)^2}\cdot 3t^2\Bigg|_{t=1} = \frac{-4\cdot 5 - 2\cdot 2 - 2\cdot 3}{(2^2+1+1)^2} = \frac{-30}{36} = \frac{-5}{6}$$

19. $V = 900\ cm^3$, $\frac{dV}{dt} = 10\ cm^3/min$, $T = 400K$, $\frac{dT}{dt} = 15\ K/min$

$$P = \frac{RT}{V}. \quad \frac{dP}{dt} = \frac{dP}{dV}\frac{dV}{dt} + \frac{dP}{dT}\frac{dT}{dt} = \frac{-RT}{V^2}\Bigg|_{T=400,\, V=900}\cdot 10 + \frac{R}{V}\Bigg|_{T=400,\, V=900}\cdot 15$$

$$= \frac{R}{900}\left(\frac{-400}{900}\cdot 10 + 15\right) = \frac{R}{900}(-40+15) = \frac{-R}{36}.$$

Since $R > 0$ we see that the pressure is decreasing.

20. $x = 70,\ \frac{dx}{dt} = 4.\ \ y = 300,\ \frac{dy}{dt} = -10$

$$\frac{du}{dt} = 0{,}0006x\, e^{-0{,}0003x^2 - 0{,}00000009y^2} \frac{dx}{dt} + 0{,}00000018y\, e^{-0{,}0003x^2 - 0{,}00000009y^2} \frac{dy}{dt} =$$

$$= e^{-0{,}0003x^2 - 0{,}00000009y^2}\left(0{,}0006x \frac{dx}{dt} + 0{,}00000018y \frac{dy}{dt}\right) =$$

$$= e^{-1{,}47 - 0{,}0081}\left(0{,}168 - 0{,}000054\right) = e^{-1{,}4781} \cdot 0{,}167946 \approx$$

$\approx 0{,}0383$, Thus, the utility is increasing.

21. $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{3x^2y^2z + y}{x^3y^2 - 3z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{2x^3yz + x}{x^3y^2 - 3z^2}$$

22. $$\frac{\partial z}{\partial x} = -\frac{\sin\left(\frac{y}{z}\right) - \frac{z}{y}\sin\left(\frac{x}{y}\right)}{-\frac{xy}{z^2}\cos\left(\frac{y}{z}\right) + \cos\left(\frac{x}{y}\right)}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{x}{z}\cos\left(\frac{y}{z}\right) + \frac{xz}{y^2}\sin\left(\frac{x}{y}\right) - 1}{-\frac{xy}{z^2}\cos\left(\frac{y}{z}\right) + \cos\left(\frac{x}{y}\right)}$$

23. $$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}\right)\left(-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}\right)\left(-\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}\right) = (-1)^3 = -1$$

24. Let $f(x,y,z) = x^4 + y^4 + z^4 - 1$. Then $S \equiv f(x,y,z) = 0$.
Thus $\frac{\partial z}{\partial x}\Big|_S = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{4x^3}{4z^3} = -\left(\frac{x}{z}\right)^3$. Similarly, $\frac{\partial z}{\partial y}\Big|_S = -\left(\frac{y}{z}\right)^3$